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Note

The depression of a graph and the diameter of its line graph

Iris Gaber-Rosenblum, Yehuda Roditty*

The Academic College of Tel-Aviv-Yaffo, Tel-Aviv, Israel

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Abstract

An edge ordering of a graph $G = (V, E)$ is an injection $f : E \to Q^+$ where Q^+ is the set of positive rational numbers. A (simple) path λ for which f increases along its edge sequence is an f-ascent, and a maximal f-ascent if it is not contained in a longer f-ascent. The depression $\varepsilon(G)$ of G is the least integer k such that every edge ordering of G has a maximal ascent of length at most k.

It has been shown in [E.J. Cockayne, G. Geldenhuys, P.J.P. Grobler, C.M. Mynhardt, J. van Vuuren, The depression of a graph, Utilitas Math. 69 (2006) 143–160] that the difference $\mathrm{diam}(L(G)) - \varepsilon(G)$ may be made arbitrarily large. We prove that the difference $\varepsilon(G) - \mathrm{diam}(L(G))$ can also be arbitrarily large, thus answering a question raised in [E.J. Cockayne, G. Geldenhuys, P.J.P. Grobler, C.M. Mynhardt, J. van Vuuren, The depression of a graph, Utilitas Math. 69 (2006) 143–160].

Keywords: Edge ordering; Increasing path; Monotone path; Depression

1. Introduction

Let $G = (V, E)$ be a graph. An *edge ordering* of G is an injection $f : E \to Q^+$ where Q^+ is the set of positive rational numbers. Denote the set of all edge orderings of G by $\mathcal{F}(G)$. A path λ in G for which $f \in \mathcal{F}(G)$ increases along its edge sequence is called an *f-ascent* (or simply *ascent* if the edge ordering is clear), and if λ has length k, it will also be called a *(k, f)-ascent*. If the path λ with vertex sequence $v_0, v_1, \ldots, v_k$ forms an f-ascent, we denote this fact by writing λ as $v_0 v_1 \ldots v_k$. An f-ascent is called *maximal* if it is not contained in a longer f-ascent. Let $h(f)$ denote the length of a shortest maximal f-ascent and define the *depression* $\varepsilon(G)$ of G by

$$\varepsilon(G) = \max_{f \in \mathcal{F}(G)} \{h(f)\},$$

that is, $\varepsilon(G)$ is the smallest integer k such that every edge ordering of G has a maximal ascent of length at most k. Therefore $\varepsilon(G) = k$ if and only if

(a) each edge ordering of G has a maximal ascent of length at most k, i.e. $\varepsilon(G) \le k$, and
(b) there exists an edge ordering f of G with no maximal ascents of length less than k, i.e. $\varepsilon(G) \ge k$.

* Corresponding author.
E-mail address: jr@post.tau.ac.il (Y. Roditty).

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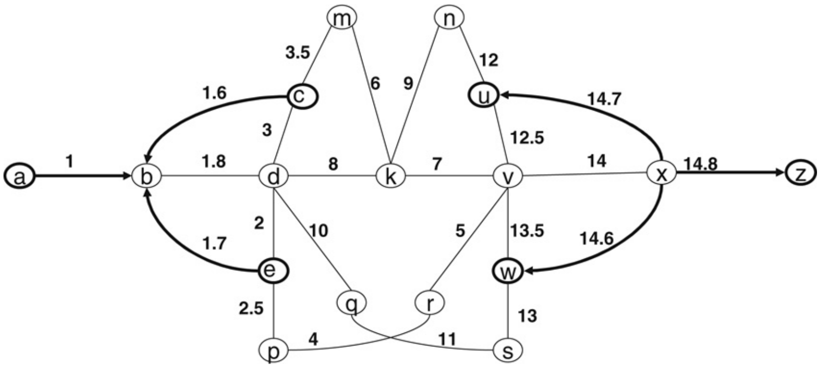

Fig. 1. The graph G.

The study of the lengths of monotone paths in edge-ordered graphs was initiated by Chvátal and Komlós [1] who posed the problem of determining the *altitude* $\alpha(G)$, the greatest integer k such that G has a (k, f)-ascent for each edge ordering $f \in \mathcal{F}(G)$, for $G = K_n$. The depression of a graph was first defined in [2], where they discussed, amongst other results, the ratio between the depression of a graph and the diameter of its line graph.

It is easy to find infinite classes of graphs G for which $\varepsilon(G) \leq \mathrm{diam}(L(G)) + 1$, where $\mathrm{diam}(L(G))$ denotes the diameter of the line graph, $L(G)$, of G.

In [2] they proved the following result:

Proposition 1 ([2]). *If a graph G has a vertex adjacent to two leaves, or to two adjacent vertices of degree* 2*, then* $\varepsilon(G) = 2$.

Clearly a graph can be made with $\mathrm{diam}(L(G))$ as large as we want and have that property.

Another related result proved in [2] is:

Proposition 2 ([2]).

- *If* $\mathrm{diam}(L(G)) = 1$*, then* $\varepsilon(G) = \mathrm{diam}(L(G)) + 1$.
- *If* $\mathrm{diam}(L(G)) = 2$*, then* $\varepsilon(G) \leq \mathrm{diam}(L(G)) + 1$.

However, there are graphs for which $\varepsilon(G) > \mathrm{diam}(L(G)) + 1$; one such example appears in [2]. The graph that they call G_2 has $\mathrm{diam}(L(G_2)) = 3$ and $\varepsilon(G) \geq 5$. In the next section we present a graph which can be extended to a class of infinite graphs for which $\varepsilon(G) > \mathrm{diam}(L(G)) + 1$. Moreover, the difference $\varepsilon(G) - \mathrm{diam}(L(G))$ can be as large as we want. This answers an open problem raised by Cockayne, Geldenhuys, Grobler, Mynhardt and van Vuuren in [2].

2. Graphs G for which $\varepsilon(G) > \mathrm{diam}(L(G)) + 1$

We begin by presenting a graph G and an edge ordering f on it such that $\mathrm{diam}(L(G)) = 5$ and $h(f) = 7$ which proves that $\varepsilon(G) \geq 7$ (see Fig. 1). We denote by $(i, j) \in E$ an edge such that its direction in a given maximal f-ascent path is from i to j. The edge ordering f on the graph G has the following property:

Proposition 3. *Let* $G = (V, E)$ *and the ordering* f *be as in Fig. 1.*

- *Any maximal* f*-ascent path starts with one of the edges* (a, b)*,* (c, b) *or* (e, b).
- *Any maximal* f*-ascent path ends at one of the edges* (x, w)*,* (x, u) *or* (x, z).

Proof. An edge (i, j) cannot start a maximal f-ascent path if one of the following conditions holds:

1. All edges that are adjacent to j are labeled with a lower value than $f(i, j)$ (or there are no edges adjacent to j but (i, j)).
2. There is an edge (k, i) such that $f(k, i) < f(i, j)$ and all edges adjacent to k are labeled with a lower value than $f(i, j)$ (or there are no edges adjacent to k but (k, i)).

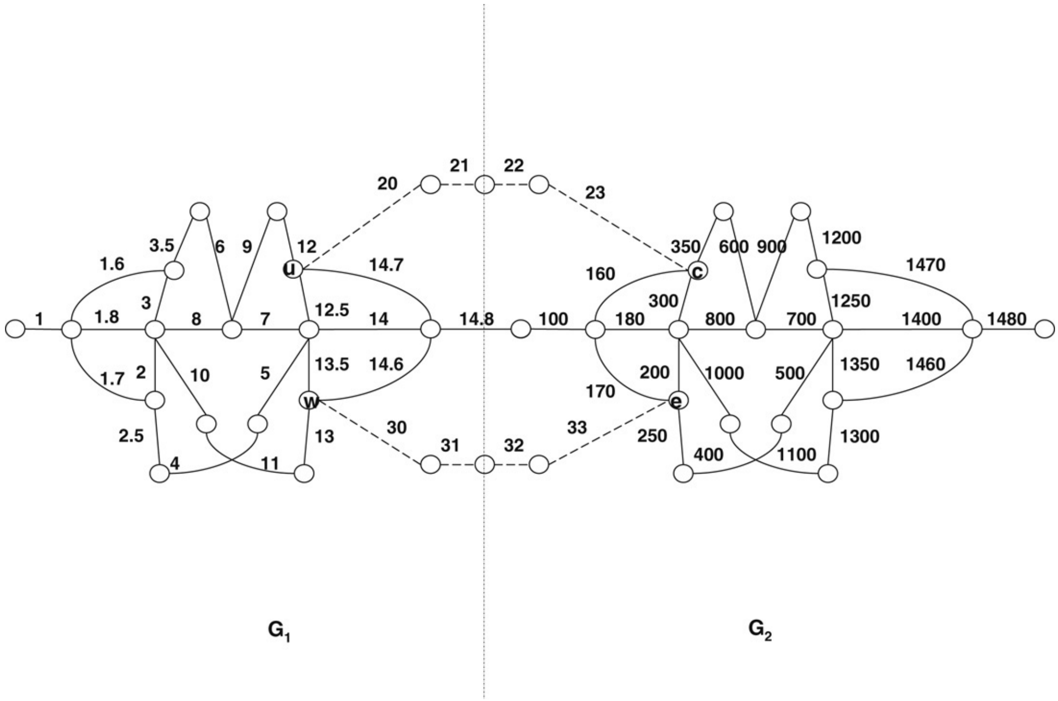

Fig. 2. The graph B_2.

The second claim is true because we can always enlarge the path that starts with (i, j) with the edge (k, i) since the path does not contain the vertex k.

For each edge of G besides (a, b), (c, b) and (e, b) we prove that it cannot start a maximal f-ascent path (see Fig. 3).

Fig. 3 is a table that specifies for each edge why it cannot start a maximal f-ascent path in the following manner. For an edge that cannot start a maximal f-ascent path for the first reason we write 'direction' in the reason column and for an edge that cannot start a maximal f-ascent path for the second reason we write a name of an edge that prohibits it from starting a maximal f-ascent path.

The proof of the second claim is very similar and therefore we omit it. ∎

Since every f-ascent path that starts with either (a, b), (c, b) or (e, b) and ends with either (x, w), (x, u) or (x, z) is of length at least 7, Proposition 3 gives the following property of the ordering f:

Corollary 4. *Every maximal f-ascent path is of length at least* 7.

Our next stage is to construct a new family of graphs, $\mathcal{B}$, as described hereafter. B_i consists of i copies of G denoted $G_1, G_2, \ldots, G_i$. For each $G_j, 1 \le j \le i$ we use the notation $V(G_j) = \{a_j, b_j, c_j, d_j, e_j, k_j, m_j, n_j, p_j, q_j, r_j, s_j, u_j, v_j, w_j, x_j, z_j\}$.

The graphs G_j are connected in the following manner. $B_1 \in \mathcal{B}$ is simply G_1, and for each $i > 1$ we construct B_{i+1} from B_i by taking the graphs B_i and G_{i+1} and connect the vertex c_i to a path of three new vertices where the rightmost new vertex is connected to the vertex u_{i+1}. The same procedure is carried out with the vertex e_i which starts a path of three new vertices that connect at the end to vertex w_{i+1}. Finally, we unify vertex z_i with the vertex a_{i+1}. Fig. 2 shows the graph B_2.

The path that connects c_j and u_{j+1} is denoted as Pup_j and the path that connects e_j and w_{j+1} is denoted as Pdown_j.

The diameter of the line graph of B_i, $i \ge 1$, is clearly

$$\mathrm{diam}(L(B_i)) = 5 + 6(i - 1).$$

The next theorem is our main result:

edge	reason
(b,a)	direction
(b,c)	(a,b)
(b,e)	(a,b)
(b,d)	(a,b)
(d,b)	direction
(d,e)	(b,d)
(e,d)	(b,e)
(d,c)	(b,d)
(c,d)	(b,c)
(e,p)	(b,e)

edge	reason
(p,e)	direction
(c,m)	(b,c)
(m,c)	direction
(p,r)	(e,p)
(r,p)	direction
(r,v)	(p,r)
(v,r)	direction
(m,k)	(c.m)
(k,m)	direction
(k,v)	(m,k)

Fig. 3. Table of edges that cannot start a maximal f-ascent path in G.

Theorem 5. *For every integer $n \geq 1$ there is a graph B for which $\varepsilon(B) - \mathrm{diam}(L(B)) > n$.*

Proof. The proof uses the graphs in $\mathcal{B}$ and the edge ordering f on G as described in Fig. 1. We prove the theorem by presenting an edge ordering called b for the graph B_i, $i > 1$. Every edge $l \in G_j$ is labeled with the value $f(l) \times 10^{2(j-1)}$. The edges of the path Pup_j are labeled $(20 \times 10^{2(j-1)}, 21 \times 10^{2(j-1)}, 22 \times 10^{2(j-1)}, 23 \times 10^{2(j-1)})$ and the edges of the path Pdown_j are labeled $(30 \times 10^{2(j-1)}, 31 \times 10^{2(j-1)}, 32 \times 10^{2(j-1)}, 33 \times 10^{2(j-1)})$.

Fig. 2 shows B_2 and the edge ordering b on it.

We claim that the edge ordering b satisfies

$$h(b) \geq 7 \times i.$$

We start by proving the claim for the graph B_2. Every maximal b-ascent path in B_2 must start with either a_1, c_1 or e_1 and end with either u_2, w_2 or z_2 according to Proposition 3. There are two possible cases:

1. The path does not pass through Pup_1 or Pdown_1.
2. The path passes through Pup_1 or through Pdown_1.

In the first case the length of the path is at least $7 + 7 = 14$ and in the second case it is at least $5 + 4 + 5 = 14$ because the shortest maximal ascent path in B_2 from a_1, c_1 or e_1 to u_1 or w_1 is of length 5, and the same holds for a shortest maximal ascent path from c_2 or e_2 to u_2, w_2 or z_2.

Consider now the graph B_i for some $i > 2$ and a maximal b-ascent path in it, P_i. The path P_i must start with either a_1, c_1 or e_1 and end with either u_i, w_i or z_i according to Proposition 3. Also, P_i must pass through all graphs G_j, $1 \leq j \leq i$.

We divide the path P_i into segments, P_j, $1 \leq j \leq i$, such that P_j uses only edges of G_j, the first two edges of Pup_j, the first two edges of Pdown_j, the last two edges of Pup_{j-1} and the last two edges of Pdown_{j-1}.

For every P_j, $1 < j < i$, there are four possible cases:

1. P_i does not pass through Pup_j, Pdown_j, Pup_{j-1} or Pdown_{j-1}.
2. P_i does not pass through Pup_j or Pdown_j, but it passes through Pup_{j-1} or Pdown_{j-1}.
3. P_i passes through Pup_j or Pdown_j, but it does not pass through Pup_{j-1} or Pdown_{j-1}.
4. P_i passes through Pup_j or Pdown_j and also through Pup_{j-1} or Pdown_{j-1}.

In the first case the length of P_j is at least 7. In the second and third cases the length of P_j is at least $5 + 2$ where the 5 is the number of edges in G_j, and in the fourth case the length of P_j is at lease $2 + 4 + 2$ because the length of the path from c_j or e_j to u_j or w_j is 4. ∎

References

[1] V. Chvátal, J. Komlós, Some combinatorial theorems on monotonicity, Canad. Math. Bull. 14 (1971) 151–157.
[2] E.J. Cockayne, G. Geldenhuys, P.J.P. Grobler, C.M. Mynhardt, J. van Vuuren, The depression of a graph, Util. Math. 69 (2006) 143–160.